HYPER VERSION

Received: *December 9, 1998,* Accepted: *December 10, 1998*

AdS/CFT correspondence and topological field theory

Edward Witten

School of Natural Sciences, Institute for Advanced Study
Olden Lane, Princeton, NJ 08540, USA
E-mail: witten@ias.edu

Abstract: In $\mathcal{N} = 4$ super Yang-Mills theory on a four-manifold M, one can specify a discrete magnetic flux valued in $H^2(M, \mathbb{Z}_N)$. This flux is encoded in the AdS/CFT correspondence in terms of a five-dimensional topological field theory with Chern-Simons action. A similar topological field theory in seven dimensions governs the space of "conformal blocks" of the six-dimensional $(0, 2)$ conformal field theory.

Keywords: Topological Field Theories, 1/N Expansion, String Duality.

JHEP12(1998)012

Contents

JHEP12(1998)012

1. Introduction

The AdS/CFT correspondence [1] relates $\mathcal{N}=4$ super Yang-Mills theory on $\mathbf{S}^4$, with gauge group $SU(N)$, to Type IIB superstring theory on $AdS_5 \times \mathbf{S}^5$, with N units of five-form flux on $\mathbf{S}^5$. The correspondence expresses gauge theory correlation functions in terms of the dependence of the string theory on the behavior near the conformal boundary of AdS_5 [2, 3].

What happens if we replace $\mathbf{S}^4$ with a more general four-dimensional spin manifold M? (M must be spin since the $\mathcal{N}=4$ theory contains spinor fields. In addition, M must be endowed with a metric that — after a suitable conformal rescaling — has positive scalar curvature, or the $\mathcal{N}=4$ theory is unstable.[1]) The behavior of the gauge theory on a four-manifold M is believed to be described, roughly speaking, in terms of the behavior of the string theory on $X \times \mathbf{S}^5$, where X is a negatively curved

[1] The instability arises because of possible runaway behavior of the massless scalars ϕ of the theory. Positivity of the scalar curvature R of M suppresses the instability because of the $R\ \mathrm{tr}\,\phi^2$ coupling required by conformal invariance.

Einstein manifold with M as conformal boundary, and one must sum over all possible choices of X. This description is somewhat rough since, in general, one might need to include branes or stringy singularities on $X \times \mathbf{S}^5$ or more general topologies that are not simply the product of $\mathbf{S}^5$ with some five-manifold. The general prescription is really that one sums over Type IIB spacetimes that near infinity look like $X \times \mathbf{S}^5$, with X a five-manifold of conformal boundary M.

The gauge theory on M that is described in this correspondence has a gauge group that is locally $SU(N)$, rather than $U(N)$. Many arguments show this, beginning with the fact that in $\mathcal{N} = 4$ super Yang-Mills theory of $U(N)$, the $U(1)$ fields would be free, while in string theory on $AdS_5 \times \mathbf{S}^5$, everything couples to gravity and no field is free. As the gauge group is $SU(N)$, its center is $\mathbb{Z}_N$.

One consequence of the fact that the gauge group is $SU(N)$ rather than $U(N)$ is that it is possible to make a baryon vertex linking N external quarks. This can be constructed using a wrapped fivebrane [4] and can also be understood [5] via a certain Chern-Simons interaction that we will describe shortly.

Our interest in the present paper will, however, be in properties that depend on the global structure of M. In $\mathcal{N} = 4$ super Yang-Mills theory, all fields are in the adjoint representation and hence, as a local gauge transformation, the center of the gauge group acts trivially. It follows that locally the gauge group could be $SU(N)/\mathbb{Z}_N$. This has two important consequences depending on the topology of M:

1. If $H^1(M, \mathbb{Z}_N) \neq 0$, it is possible to consider gauge transformations on M that are not single-valued in $SU(N)$ but are single-valued in $SU(N)/\mathbb{Z}_N$. For example, if $M = \mathbf{S}^1 \times Y$, one can consider a gauge transformation that is constant in the Y direction and whose restriction to the $\mathbf{S}^1$ direction determines any element of $\pi_1(SU(N)/\mathbb{Z}_N) = \mathbb{Z}_N$. This gives an important group $F \cong \mathbb{Z}_N$ of global symmetries of the thermal physics on Y. More generally, $F = H^1(M, \mathbb{Z}_N)$ classifies $SU(N)/\mathbb{Z}_N$ gauge transformations that cannot be lifted to $SU(N)$.

2. If $H^2(M, \mathbb{Z}_N) \neq 0$, it is possible to consider $SU(N)/\mathbb{Z}_N$ bundles with "discrete magnetic flux" [6]. In fact, $SU(N)/\mathbb{Z}_N$ bundles on M are classified topologically by specifying the instanton number and also a characteristic class w that takes values in $K = H^2(M, \mathbb{Z}_N)$. (For example, if $N = 2$, we have $SU(2)/\mathbb{Z}_2 = SO(3)$, and w coincides with the second Stieffel-Whitney class of the gauge bundle.)

How to see in the AdS/CFT correspondence the thermal symmetry F has been discussed elsewhere [7]. For our present purposes, we need to recall just one point from that discussion. Type IIB superstring theory has two two-form fields B_{NS} and B_{RR}. In compactification on $X \times \mathbf{S}^5$ (with N units of five-form flux on $\mathbf{S}^5$), one

JHEP12(1998)012

gets a low energy effective action on X that contains a Chern-Simons term

$$L_{CS} = N I_{CS}\,, \tag{1.1}$$

where

$$I_{CS} = -\frac{i}{2\pi}\int_X B_{RR}\wedge dB_{NS} \tag{1.2}$$

is the basic Chern-Simons invariant of the two B-fields. (The action L_{CS} is important in one approach to the baryon vertex [5].) Though the integrand in I_{CS} is not gauge-invariant, I_{CS} is gauge-invariant mod $2\pi i$ on a closed five-manifold X. It can be written more invariantly as follows: if X is the boundary of a six-manifold Y over which the two B-fields extend, and we write the field strength of a B-field as $H = dB$, then one can write I_{CS} in a manifestly gauge-invariant way as

$$I_{CS} = -\frac{i}{2\pi}\int_Y H_{RR}\wedge H_{NS}\,. \tag{1.3}$$

More generally, if Y does not exist, a more subtle approach is needed to define I_{CS}. For a general Chern-Simons interaction, one can follow a slightly abstract approach explained in section 2.10 and the end of the introduction to section 4 in [8]. For the particular Chern-Simons theory that we are discussing here, one can define I_{CS} as a cup product in Cheeger-Simons cohomology.[2]

A theory of the two B-fields with Lagrangian precisely 1.1 is a simple but subtle topological field theory, of a sort first considered in [10]. The actual low energy effective action that arises in compactification of Type IIB superstring theory on $\mathbf{S}^5$ has many couplings beyond the Chern-Simons interaction; the additional couplings have however a larger number of derivatives and so are irrelevant for certain questions.

The goal of the present paper is to understand how to encode in terms of the AdS/CFT correspondence the dependence of the gauge theory on the discrete magnetic flux w. As we will see, the topological field theory with Lagrangian 1.1 plays a starring role in the analysis. Roughly speaking, what in the gauge theory description appears as the discrete magnetic flux w appears in the gravitational description on $X\times\mathbf{S}^5$ as a quantum state of this topological field theory. We state a precise

[2] In the language of Cheeger and Simons [9], a B-field on a manifold X is an element of $\widehat{H}^2(X,U(1))$, the group of differential two-characters on X with values in $U(1)$. For two elements $B_{RR}, B_{NS}\in\widehat{H}^2(X,U(1))$, Cheeger and Simons describe in Theorem 1.11 a product $B_{RR}*B_{NS}\in\widehat{H}^5(X,U(1))$. For X of dimension five, $\widehat{H}^5(X,U(1)) = U(1)$, and we set $\exp\left(-I_{CS}(B_{RR},B_{NS})\right) = B_{RR}*B_{NS}$. Equation 1.15 in [9] asserts that if B_{RR} is topologically trivial and hence can be defined by an ordinary two-form, I_{CS} can be computed by the integral written in 1.2: $I_{CS} = -(i/2\pi)\int_X B_{RR}\wedge H_{NS}$. Of course, there is a similar formula $I_{CS} = (i/2\pi)\int_X H_{RR}\wedge B_{NS}$ if B_{NS} is topologically trivial. It can also be proved that $B_{RR}*B_{NS}$ can be defined by the formula in the text if X is the boundary of an appropriate six-manifold Y. The usefulness of interpreting the action in terms of differential characters was explained to me by M. Hopkins, who also pointed out facts that we will use in section 3.4.

JHEP12(1998)012

conjecture in section 2 and carry out some computations supporting it in section 3. Then in section 4, we make an extension to the $(2,0)$ superconformal field theory in six dimensions, and in section 5 we discuss 't Hooft and Wilson loops in the four-dimensional $\mathcal{N}=4$ theory.

Since we will be developing a fairly elaborate theory to understand in terms of gravity the dependence of the gauge theory on the discrete magnetic flux, it is reasonable to ask what examples are known to which this theory can be applied. Actually, as we noted at the outset, there are relatively few M's for which the AdS/CFT correspondence is expected to work (M must be spin and with positive scalar curvature), and there are presently very few such examples for which anything is known about the possible five-manifolds X with conformal boundary M. One simple but important example, in which one can verify the importance of summing over different choices of X, is the case $M = \mathbf{S}^1 \times \mathbf{S}^3$. There are [11] two known choices of X, namely $X_1 = \mathbf{S}^1 \times B_4$ and $X_2 = B_2 \times \mathbf{S}^3$ (here B_n denotes an n-dimensional ball); many important properties of Yang-Mills theory at nonzero temperature are reflected in the behavior of these two X's [12].

This example has $H^2(M, \mathbb{Z}_n) = 0$ and so does not serve as a good illustration of the issues explored in the present paper. But one can readily modify it to give an example with nontrivial discrete magnetic fluxes. Identify $\mathbf{S}^3$ with the $SU(2)$ manifold and let H be a discrete subgroup of $SU(2)$, acting on the right. Set $M_H = \mathbf{S}^1 \times \mathbf{S}^3/H$. M_H is the boundary of $X_{i,H} = X_i/H$. For suitable H, $H^2(M_H, \mathbb{Z}_N) \neq 0$, so this gives a simple example to which our theory can be applied, showing in particular that the theory is nonvacuous.

This example actually has the following interesting property. $X_{2,H}$ has orbifold singularities (because H acts freely on $\mathbf{S}^3$ but not on B_4). The orbifold singularities are harmless in string theory and will be resolved as one varies the metric on M_H. The manifold $X_{2,H}$ will lose and acquire singularities and undergo monodromies as the metric on M_H is varied.

Another example is $M = \mathbf{S}^2 \times \mathbf{S}^2$, where some X's have been constructed in [13] and investigated independently in [14]. This example has the property that $H^2(M, \mathbb{Z})$ is not a torsion group.

Some technical details. We conclude this introduction by summarizing a few useful details. In this paper, the symbol b_i will denote the i^{th} Betti number of the four-manifold M. We also write the order of the finite group $H^i(M, \mathbb{Z}_N)$ as $N^{\widetilde{b}_i}$. If there is no torsion in the integral cohomology of M, then $b_i = \widetilde{b}_i$ (In general the $\widetilde{b}_i$ are not integers). Using Poincaré and Pontryagin duality, it is possible to prove that $b_{4-i} = b_i$ and likewise $\widetilde{b}_{4-i} = \widetilde{b}_i$. The last statement arises because there is a nondegenerate pairing $H^i(M, \mathbb{Z}_N) \times H^{4-i}(M, \mathbb{Z}_N) \to \mathbb{Z}_N$ (given by the cup product); we write the product of $x \in H^i(M, \mathbb{Z}_N)$ with $y \in H^{4-i}(M, \mathbb{Z}_N)$ as $x \cdot y$. Nondegeneracy of the pairing implies that these groups are of the same order (and

JHEP12(1998)012

in fact are isomorphic as abelian groups). One also has $b_0 = \widetilde{b}_0 = 1$. We write χ and σ for the Euler characteristic and signature of M; we recall the definition $\chi = \sum_{i=1}^{4}(-1)^i b_i$. Using the long exact sequence of cohomology groups derived from the short exact sequence of groups $0 \rightarrow \mathbb{Z} \xrightarrow{N} \mathbb{Z} \rightarrow \mathbb{Z}_N \rightarrow 0$ (the first map is multiplication by N and the second is reduction modulo N), one can prove that in fact

$$\sum_{i=0}^{4}(-1)^i b_i = \sum_{i=0}^{4}(-1)^i \widetilde{b}_i\,. \tag{1.4}$$

Hence $b_1 - \frac{1}{2}b_2 = \widetilde{b}_1 - \frac{1}{2}\widetilde{b}_2$, a fact we will use later.

2. Role of the topological field theory

2.1 The problem

As a prelude to our main subject, let us ask: In studying the $\mathcal{N} = 4$ super Yang-Mills theory on a four-manifold M via the AdS/CFT correspondence, do we expect to see Montonen-Olive S-duality? The answer is, "Not in a naive way," since $SU(N)$ is not a self-dual group. For example, if

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \tag{2.1}$$

is the coupling parameter of the theory, then under $\tau \rightarrow -1/\tau$ we expect $SU(N)$ to be exchanged with $SU(N)/\mathbb{Z}_N$.

The $SU(N)$ and $SU(N)/\mathbb{Z}_N$ theories are equivalent locally, and they are essentially equivalent on a four-manifold such as $M = \mathbf{S}^4$ or $\mathbf{S}^3 \times \mathbf{S}^1$ whose second cohomology group vanishes.[3] These two theories really differ in an interesting way when $H^2(M, \mathbb{Z}_N) \neq 0$. The reason is that $SU(N)$ bundles on M are classified just by the instanton number, but $SU(N)/\mathbb{Z}_N$ bundles are classified by an additional topological invariant which is the "discrete magnetic flux" $w \in H^2(M, \mathbb{Z}_N)$ mentioned in the introduction. The $SU(N)$ theory has a partition function $Z(\tau)$ (we suppress M from the notation when this is likely to cause no confusion), but the $SU(N)/\mathbb{Z}_N$ theory has a family of partition functions $Z_w(\tau)$, one for each $w \in H^2(M, \mathbb{Z}_N)$. The

[3] On such a manifold, the $SU(N)$ and $SU(N)/\mathbb{Z}_N$ theories have the same correlation functions; their partition functions differ by an elementary factor described in eqn. (3.17) of [15]. This factor arises because the groups of $SU(N)$ and $SU(N)/\mathbb{Z}_N$ gauge transformations differ slightly; the center of $SU(N)$ consists of gauge transformations in $SU(N)$ that are not considered in $SU(N)/\mathbb{Z}_N$, while $F = H^1(M, \mathbb{Z}_N)$ classifies the classes of global $SU(N)/\mathbb{Z}_N$ gauge transformations that cannot be lifted to $SU(N)$. With $N^{\widetilde{b}_1}$ the order of the finite group F, the group of $SU(N)$ gauge transformations has volume $N^{1-\widetilde{b}_1}$ times that for $SU(N)/\mathbb{Z}_N$, and hence the $SU(N)$ partition function is $N^{-1+\widetilde{b}_1}$ times that for $SU(N)/\mathbb{Z}_N$, a fact which is incorporated in the next equation in the text. If, as assumed in [15], there is no torsion in the cohomology of M, then $\widetilde{b}_1 = b_1$, and the factor becomes N^{-1+b_1}, as written in [15].

JHEP12(1998)012

relation between the two is that the $SU(N)$ partition function is obtained from the $SU(N)/\mathbb{Z}_N$ partition function by setting $w = 0$, up to an elementary factor (mentioned in the footnote):

$$Z_{SU(N)}(\tau) = N^{-1+\widetilde{b}_1} Z_0(\tau)\,. \tag{2.2}$$

The $SU(N)$ theory is thus obtained by setting $w = 0$, but in the $SU(N)/\mathbb{Z}_N$ theory, one sums over all w. Since these operations are supposed to be exchanged under $\tau \to -1/\tau$, clearly the contribution with a given w cannot be $SL(2,\mathbb{Z})$-invariant.

Generalizing ideas of 't Hooft [6], it has been argued [15] that the $Z_w(\tau)$ transform as a unitary representation of $SL(2,\mathbb{Z})$. To describe this representation, one must use the fact that the cup product gives a natural pairing $H^2(M,\mathbb{Z}_N)\times H^2(M,\mathbb{Z}_N) \to H^4(M,\mathbb{Z}_N) = \mathbb{Z}_N$, as mentioned at the end of the introduction. The most interesting part of the action of $SL(2,\mathbb{Z})$ is the behavior under $\tau \to -1/\tau$, which according to [15] is a sort of discrete Fourier transform:

$$Z_v(-1/\tau) = N^{-\widetilde{b}_2/2}\left(\frac{\tau}{i}\right)^{W/2}\left(\frac{\overline{\tau}}{-i}\right)^{\overline{W}/2} \sum_{w\in H^2(M,\mathbb{Z}_N)} \exp(2\pi i v\cdot w/N)\, Z_w(\tau)\,. \tag{2.3}$$

The intuitive idea of this formula is that $\tau \to -1/\tau$ exchanges magnetic and electric flux, but the discrete electric flux is defined by a Fourier transform with respect to the magnetic variable. The factor of $N^{-\widetilde{b}_2/2}$ is needed for $S^2 = 1$, since the order of the finite group $K = H^2(M,\mathbb{Z}_N)$ is $N^{\widetilde{b}_2}$. (In [15], the cohomology was assumed to be torsion-free, and this factor reduced to N^{b_2}.) The modular weights W and $\overline{W}$ are expected to be linear functions of χ and σ (the Euler characteristic and signature of M). They cannot be determined just from gauge theory, as they can be modified by adding gravitational couplings of the general form $f(\tau,\overline{\tau})RR$ (R being the Riemann tensor of M); to predict the modular weights that are observed in computing the $Z_v(\tau)$ using the AdS/CFT correspondence, one would need to know precisely which such couplings are determined by this correspondence. (In [15], $\overline{W}$ was set to zero, since a twisted version of the theory with a holomorphically varying partition function was considered.)

In addition to 2.3, the $SL(2,\mathbb{Z})$ transformation law of the $Z_v(\tau)$ is specified by describing the behavior under $\tau \to \tau+1$. This is described in eqn. (3.14) of [15] and is determined by the fact that the instanton number of an $SU(N)/\mathbb{Z}_N$ bundle is not an integer but is of the form (if M is a spin manifold)

$$\frac{v^2}{2N}\ \text{modulo}\ \mathbb{Z}\,, \tag{2.4}$$

as in equation (3.13) of [15]. (A term $v^2/2$, which is integral if M is spin, has been omitted.) The transformation $\tau \to \tau+1$ amounts to $\theta \to \theta + 2\pi$ in gauge theory, so the transformation law is

$$Z_v(\tau+1) = \exp\left(2\pi i(v^2/2N - s)\right) Z_v(\tau)\,, \tag{2.5}$$

JHEP12(1998)012

where s is a constant that reflects the fact that the gravitational couplings $f(\tau,\overline{\tau})RR$ may not be invariant under $\tau \to \tau + 1$.

It follows from 2.2 and 2.3 that, essentially as in eqn. (3.18) of [15], the $SU(N)$ and $SU(N)/\mathbb{Z}_N$ partition functions are related by

$$Z_{SU(N)}(-1/\tau) = N^{-\chi/2} \left(\frac{\tau}{i}\right)^{W/2} \left(\frac{\overline{\tau}}{-i}\right)^{\overline{W}/2} Z_{SU(N)/\mathbb{Z}_N}(\tau)\,. \tag{2.6}$$

This is essentially the Montonen-Olive formula, saying that $\tau \to -1/\tau$ exchanges $SU(N)$ and $SU(N)/\mathbb{Z}_N$; the prefactors reflect gravitational couplings not present in the original Montonen-Olive formulation[4] on flat $\mathbf{R}^4$.

2.2 The partition function as a vector in Hilbert space

Now we will make a change in viewpoint, which is suggested by experience with rational conformal field theory in two dimensions. A rational conformal field theory on a Riemann surface Σ of positive genus generally has, if one considers the chiral degrees of freedom only, not a single partition function, but a collection of partition functions. It is useful [16] to group these together as a vector in a Hilbert space, which one can think about using quantum mechanical intuition [17], and which one can ultimately understand using topological field theory in one dimension higher [18].

So we introduce a Hilbert space $\mathcal{H}$ with one orthonormal basis vector Ψ_w for every $w \in H^2(M,\mathbb{Z}_N)$. We regard the $Z_w(\tau)$ as components of a vector $\Psi(\tau) \in \mathcal{H}$, with $\Psi(\tau) = \sum_w Z_w(\tau)\Psi_w$. Thus

$$Z_w(\tau) = (\Psi_w, \Psi(\tau))\,. \tag{2.7}$$

Since the gauge theory "partition function" is thus not an ordinary function but takes values in the Hilbert space $\mathcal{H}$, the AdS/CFT correspondence can only make sense if it is similarly true that the Type IIB partition function on a manifold such as $X \times \mathbf{S}^5$ with conformal boundary M is not an ordinary function but takes values in $\mathcal{H}$.

How can this be? At this point, we must recall the general structure of the AdS/CFT correspondence.

A very general class of observables in quantum field theory is the following. Let the $\mathcal{O}_i$ be a basis for the space of local gauge-invariant operators, and let J_i be c-number sources that couple to them. Thus the generating functional of correlation functions is

$$Z(\tau; J_i) = \left\langle \exp\left(\sum_i \int_M J_i\mathcal{O}_i\right)\right\rangle, \tag{2.8}$$

[4]The derivation of 2.6 in [15] assumed no torsion in $H^2(M,\mathbb{Z}_N)$ and gave for the first factor $N^{-1+b_1-b_2/2} = N^{-\chi/2}$. More generally, including the torsion, one gets $N^{-1+\widetilde{b}_1-\widetilde{b}_2/2}$, but as explained at the end of the introduction, these two factors are equal.

JHEP12(1998)012

where $\langle\ \ \rangle$ denotes the (unnormalized) expectation value. The usual claim in the AdS/CFT correspondence is that to compute via string theory on $X \times \mathbf{S}^5$ this generating functional in the conformal field theory on M, one must fix the values of fields on X – near its boundary – to an asymptotic behavior determined by the J_i. Hence for fixed J_i, one usually claims that the boundary behavior of the fields on $X \times \mathbf{S}^5$ are fixed; the partition function is then a "number," that is a function of the J_i. To resolve our present conundrum, we must show that if $H^2(M, \mathbb{Z}_N) \neq 0$, then even after specifying the values of all sources J_i for all local operators $\mathcal{O}_i$ on M, the boundary values of the fields on $X \times \mathbf{S}^5$ are not completely determined; and the dependence on the extra data, whatever it is, must be such that the partition function for given J_i is not a number but a vector in $\mathcal{H}$.

The reason that this is so is that fixing the behavior of all of the local gauge-invariant observables near the boundary of $X \times \mathbf{S}^5$ does not completely specify the gauge-invariant data near the boundary. There is global gauge-invariant information that cannot be measured locally, because the Type IIB theory has the two two-form fields B_{RR} and B_{NS}. One can always add to any given B-field a flat B-field, without affecting any local gauge-invariant information. Flat B-fields on a spacetime such as $Y = X \times \mathbf{S}^5$ are classified modulo gauge transformations by $H^2(Y, U(1))$. Near the boundary, this reduces to $H^2(M, U(1))$. For $H^2(M, \mathbb{Z}_N)$ to be nonzero (with M of dimension four), $H^2(M, U(1))$ must likewise be nonzero. Hence, our puzzle arises only when there are flat B-fields near M, in which case the partition function of the string theory on $X \times \mathbf{S}^5$ depends on additional data and not only on the sources J_i of gauge-invariant fields.

Roughly speaking, the additional data can be measured by "θ angles"

$$\begin{aligned} \alpha_{NS} &= \int_S B_{NS}\,, \\ \alpha_{RR} &= \int_S B_{RR}\,, \end{aligned} \tag{2.9}$$

with S a homologically nontrivial two-dimensional surface in M. One might think that the partition function of the string theory on $X \times \mathbf{S}^5$ would be a function of these theta angles, as well as the sources J_i. That would be an interesting result, but not quite what we need. We want instead to see the finite group $H^2(M, \mathbb{Z}_N)$. The reason that the string theory partition function should not be regarded simply as a function of α_{NS} and α_{RR} is that these are, in a sense, canonically conjugate variables.

The low energy effective action for B_{RR}, B_{NS}, after reduction on $\mathbf{S}^5$, looks something like

$$L = -\frac{iN}{2\pi}\int_X B_{RR} \wedge dB_{NS} + \frac{1}{2\gamma}\int_X |dB_{RR}|^2 + \frac{1}{2\gamma'}\int_X |dB_{RR}|^2 + \cdots \tag{2.10}$$

JHEP12(1998)012

Here the first term is the Chern-Simons term, whose significance for understanding the role of the center of the gauge group in the AdS/CFT correspondence was already mentioned in the introduction. The second and third terms (with constants γ, γ' that depend on τ) are the conventional kinetic terms for two-form fields. The "$\cdots$" are additional gauge-invariant terms of higher dimension.

The first term in 2.10 – the Chern-Simons term – is the important one for our present purposes, for several reasons. The obvious reason is that this is the unique term with only one derivative and hence dominates at long distances (recall that the conformal boundary of X is "infinitely far away," so the behavior near the boundary is a question of long distance physics). Perhaps even more fundamentally, the second and third terms in 2.10 and all higher terms are integrals of gauge-invariant local densities and hence insensitive to the α's, which contribute only to the first term. The Chern-Simons term is gauge-invariant but is not the integral of a gauge-invariant local density; it can and in general does change if one shifts the B-fields by a flat B-field (a fact that was crucial in [7] to enable the expected thermal symmetry of $SU(N)$ gauge theory at nonzero temperature to emerge from the AdS/CFT correspondence).

So to address the question of α-dependence of the partition function, we can use near the boundary of $X \times \mathbf{S}^5$ the simplified Chern-Simons action

$$L_{CS} = NI_{CS} = -\frac{iN}{2\pi}\int_X B_{RR} \wedge dB_{NS}\,. \tag{2.11}$$

If we write $H = dB$ for the field strength of a B-field, then the equations of motion derived from the Chern-Simons action are $H_{RR} = H_{NS} = 0$, so this action governs only the α-dependence (not the modes of nonzero H, which are massive in spacetime and have been integrated out to reduce to 2.11; their behavior near the boundary of $X \times \mathbf{S}^5$ is determined in the usual way by the sources J_i of gauge-invariant local operators on the boundary).

The Chern-Simons action L_{CS} does not depend on a metric on X, so the theory governing the α-dependence is a topological field theory, of a familiar kind [10]. From the first-order form of L_{CS}, we see that B_{RR} and B_{NS} are canonically conjugate variables in this topological field theory. After imposing the equations of motion and dividing by gauge transformations, this means that α_{RR} and α_{NS} are canonically conjugate. Hence, we should not attempt to compute the partition function as a function of both α_{RR} and α_{NS}.

What we should do instead follows from general concepts of quantum mechanics. Near M, X looks like $M \times \mathbf{R}$, with $\mathbf{R}$ the "time" direction. By quantizing the Chern-Simons theory on $M \times \mathbf{R}$, we obtain a "quantum Hilbert space" $\mathcal{H}'$ associated with M. We should interpret the partition function of the theory on X as determining, not a number, but a vector in $\mathcal{H}'$. Because $\mathcal{H}'$ is obtained by quantizing the metric-independent Chern-Simons Lagrangian, it depends only on the topology of M, and not on a metric.

JHEP12(1998)012

Moreover, the topological field theory with action L_{CS} has an $SL(2,\mathbb{Z})$ symmetry, acting on the pair

$$\begin{pmatrix} B_{RR} \\ B_{NS} \end{pmatrix} \tag{2.12}$$

in the standard fashion. (A subtlety concerning this assertion will be explained in section 3.4.) The Hilbert space $\mathcal{H}'$ hence has a natural action of $SL(2,\mathbb{Z})$.

This is almost what we need: to agree with the expectations in the boundary gauge theory, the partition function of the string theory on $X \times \mathbf{S}^5$ should be (once the gauge-invariant boundary data are specified) not a number but a vector in a Hilbert space $\mathcal{H}$. $\mathcal{H}$ is determined purely topologically by M (by definition, it has an orthonormal basis consisting of vectors Ψ_w for each $w \in H^2(M,\mathbb{Z}_N)$), and has an action of $SL(2,\mathbb{Z})$ that we described in section 2.1.

So all will be well if $\mathcal{H} = \mathcal{H}'$. Actually, we must describe somewhat more precisely in what sense these two spaces should coincide. The description of the space $\mathcal{H}$ via its basis Ψ_w, $w \in H^2(M,\mathbb{Z}_N)$ is not invariant under S-duality (which as seen in 2.3 does not preserve this basis). To make this description, we need to pick a notion of what we mean by "magnetic flux," as opposed to "electric flux." Such a choice breaks $SL(2,\mathbb{Z})$. $SL(2,\mathbb{Z})$ is broken in the same way if we introduce a two-dimensional lattice $\Lambda = \mathbb{Z}^2$ on which $SL(2,\mathbb{Z})$ acts in the standard way, and pick a "polarization" of Λ. One can think of a polarization as a choice of one direction in the lattice Λ. Alternatively, one can think of the pair $(\alpha_{RR}, \alpha_{NS})$ as taking values in $\mathbf{R}^2/\Lambda$; a polarization then is just a choice of what integral linear combination of α_{RR} and α_{NS} is the "momentum" variable. So we must show that for every choice of polarization, $\mathcal{H}'$ acquires a basis in one-to-one correspondence with the elements of $H^2(M,\mathbb{Z}_N)$, and that $SL(2,\mathbb{Z})$ acts on $\mathcal{H}'$ as it does on $\mathcal{H}$.

These assertions will be demonstrated in section 3.

3. Quantization of the topological field theory

3.1 Preliminaries

To quantize the Chern-Simons theory on $M \times \mathbf{R}$, we first work out the gauge-invariant classical phase space. We work in the gauge (analogous to $A_0 = 0$ gauge for gauge theory) in which $i\,0$ components of B_{RR} and B_{NS} (i and 0 label directions tangent to M and $\mathbf{R}$, respectively) vanish. In quantization, we restrict to time zero; together with the gauge choice, this means that B_{RR} and B_{NS} are B-fields on M. The canonical commutation relations, if written out in detail in components, read

$$[B_{RR\,ij}(x), B_{NS\,kl}(y)] = -\frac{2\pi i}{N}\epsilon_{ijkl}\delta^4(x,y)\,, \tag{3.1}$$

with $[B_{RR}(x), B_{RR}(y)] = [B_{NS}(x), B_{NS}(y)] = 0$. Here, x and y are points in M, and $i,j,k,l = 1\ldots 4$ are indices tangent to M.

JHEP12(1998)012

The "Gauss's law" constraint ($\delta L/\delta B_{i0} = 0$, where B is B_{RR} or B_{NS}) gives $H_{RR} = H_{NS} = 0$. So, modulo gauge transformations, the phase space consists of pairs of flat B-fields.

Flat B-fields are classified by $H^2(M, U(1))$. B-fields in general are classified topologically by the characteristic class $[H] \in H^3(M, \mathbb{Z})$. However, for a flat B-field, this characteristic class vanishes as a differential form, and hence represents a *torsion* element of $H^3(M, \mathbb{Z})$. Let $H^3(M, \mathbb{Z})_{tors}$ be the torsion subgroup of $H^3(M, \mathbb{Z})$. To any given flat B-field we can, without changing the topological type, add a flat and topologically trivial B-field, via a transformation $B \to B + \beta$, where β is a globally-defined, closed two-form. We should consider β trivial if its periods are multiples of 2π (since the periods of B can be shifted by multiples of 2π by a gauge transformation), so we regard β as an element of $H^2(M, \mathbf{R})/2\pi H^2(M, \mathbb{Z})$. The space $H^2(M, U(1))$ of flat B-fields thus fits into an exact sequence

$$0 \to H^2(M, \mathbf{R})/2\pi H^2(M, \mathbb{Z}) \to H^2(M, U(1)) \to H^3(M, \mathbb{Z})_{tors} \to 0\,. \tag{3.2}$$

This says that a flat B-field has a characteristic class in $H^3(M, \mathbb{Z})_{tors}$, and that two flat B-fields with the same characteristic class differ by an element of $H^2(M, \mathbf{R})/2\pi H^2(M, \mathbb{Z})$, which classifies topologically trivial flat B-fields.

Topologically trivial flat B-fields can be measured by their periods, which are the "world-sheet theta angles,"

$$\begin{aligned} \alpha_{NS} &= \textstyle\int_S B_{NS}\,, \\ \alpha_{RR} &= \textstyle\int_S B_{RR}\,, \end{aligned} \tag{3.3}$$

where S is a closed two-surface in M.

There are thus two parts of the quantization: to quantize the α's, and to take account of the finite group $H^3(M, \mathbb{Z})_{tors}$. These turn out to present quite different problems and a direct treatment of the quantization involves many subtle details. On the other hand, everything we need to know can be deduced from the symmetries of the problem. In section 3.2, we will consider this analysis using the symmetries. Then – for the benefit of curious readers – we enter into a direct analysis of the quantization.

3.2 Symmetries

The most obvious symmetry of the problem is that we can add to B_{RR} or B_{NS} any B-field B' such that NB' is pure gauge. For instance, under

$$B_{RR} \to B_{RR} + B'\,, \tag{3.4}$$

the Chern-Simons action $L_{CS}(B_{RR}, B_{NS}) = NI_{CS}(B_{RR}, B_{NS})$ changes by $L_{CS} \to L_{CS} + NI_{CS}(B', B_{NS}) = L_{CS} + I_{CS}(NB', B_{NS})$. But $I_{CS}(NB', B_{NS}) = 0$ as NB' is pure gauge.

JHEP12(1998)012

For NB' to be pure gauge means necessarily that B' is flat. Thus, B' determines an element of $H^2(M, U(1))$ that is "of order N," or in other words is annihilated by multiplication by N.

We can be more explicit about the quantum field operators that generate these symmetries. (They are analogs of operators considered many years ago in two-dimensional rational conformal field theory [17].) We construct them using the two-form counterparts of the familiar Wilson and 't Hooft loop operators of gauge theories.

First we consider the counterparts of Wilson loops. Let S and T be closed two-surfaces in M, and let

$$\begin{aligned} \Phi_{RR}(S) &= \exp\left(i\int_S B_{RR}\right), \\ \Phi_{NS}(T) &= \exp\left(i\int_T B_{NS}\right). \end{aligned} \tag{3.5}$$

Since these are gauge-invariant operators, they act on the classical phase space and map flat B-fields to flat B-fields. From the canonical commutation relations, we can deduce that

$$\Phi_{RR}(S)\Phi_{NS}(T) = \Phi_{NS}(T)\Phi_{RR}(S)\exp\left(\left(\frac{2\pi i}{N}\right) S\cdot T\right), \tag{3.6}$$

where $S \cdot T$ is the intersection number of the oriented surfaces S and T.

In particular, while $\Phi_{RR}(S)$ commutes with B_{RR}, it shifts B_{NS} by a flat B-field that is essentially determined by 3.6 to be "Poincaré dual" to S. In other words, we interpret the relation $\Phi_{RR}(S)\Phi_{NS}(T)\Phi_{RR}(S)^{-1} = \Phi_{NS}(T)\exp(2\pi i S\cdot T/N)$ to mean that conjugation by $\Phi_{RR}(S)$ shifts B_{NS} by a flat B-field with delta-function support on S (this assertion is in any case clear from the canonical commutation relations), thus multiplying $\Phi_{NS}(T)$ by a phase. The $1/N$ in the exponent in 3.6 means that Φ_{NS} is shifted by a flat B-field of order N. Conversely, conjugation by $\Phi_{NS}(T)$ shifts B_{RR} by such a field. Thus, these operators generate the symmetries that we exhibited directly in 3.4. The c-number factor in the commutation relation 3.6 shows that these symmetries do not commute with one another; there is a central extension by the N^{th} roots of unity.

The operator $\Phi_{RR}(S)$, in acting on gauge-invariant states, is trivial if S consists of N copies of another closed surface S'. That is because in such a case, $\Phi_{RR}(S) = \Phi_{RR}(S')^N$; but $\Phi_{RR}(S')$ shifts B_{NS} by a B-field of order N, and hence $\Phi_{RR}(S')^N$ shifts it by a pure gauge. Likewise, $\Phi_{NS}(T)$ is trivial if $T = NT'$ for some T'.

$\Phi_{RR}(S)$ (or $\Phi_{NS}(T)$) is also trivial if S (or T) is the boundary of a three-manifold $U \subset M$. For then

$$\Phi_{RR}(S) = \exp\left(i\int_U H_{RR}\right), \tag{3.7}$$

and this is trivial as an operator on gauge-invariant states because of the Gauss law constraint $H = 0$.

JHEP12(1998)012

So the operators $\Phi_{RR}(S)$ depend on S modulo boundaries, and are trivial if S is of the form NS'. So far we have assumed that S has no boundary, but we will next see that we can define an operator $\Phi_{RR}(S)$ with similar properties if S has a boundary, provided this boundary is of the form NC for some circle $C \subset M$. This will make possible the following simple description of the symmetry group. The group $H_2(M, \mathbb{Z}_N)$ classifies two-surfaces $S \subset M$ with boundaries of the form NC, modulo those of the form NS' and modulo boundaries. So once we show that S can have a boundary of the claimed kind, we will have shown that the operators $\Phi_{RR}(S)$ (and similarly the operators $\Phi_{NS}(T)$) are classified by $H_2(M, \mathbb{Z}_N)$. Equivalently, by Poincaré duality, they are classified by $H^2(M, \mathbb{Z}_N)$.

't Hooft loops. To see how S can have a boundary, we begin with a rather different-sounding question. Are there also in this theory symmetry operators that are analogous to the 't Hooft loops of gauge theory? This would mean the following. We consider a circle $C \subset M$. Deleting C from M, we make a "singular gauge transformation" on B_{NS} (or B_{RR}) such that on a small three-sphere that links once around C, the integral of H_{NS} equals 2π. This means that we put on C a "magnetic source" for B_{NS}.

To find such operators, we can go back to Type IIB superstring theory on $X \times \mathbf{S}^5$. In this theory, the NS fivebrane is a magnetic source for B_{NS}, so the object sought in the last paragraph would be a fivebrane wrapped on $C \times \mathbf{S}^5$. However, as explained in [4], such a fivebrane must be the boundary of N D-strings. In other words, in addition to the magnetic loop, a D-string must be present with a world-volume S whose boundary consists of N copies of C.

The operator $\Phi_{RR}(S)$ describes the coupling of B_{RR} to the D-brane worldvolume S. So we have learned that S can have a boundary of the form NC, provided that there is at the same time a magnetic source for B_{NS} on S.

The recourse to string theory to show the existence of mixed Wilson/'t Hooft operators of this kind is a convenient shortcut, but of course it would be desireable to demonstrate their existence directly in the five-dimensional topological field theory. In fact, this has essentially already been done in section 5 of [5], in the course of explaining a low energy approach to the existence of the baryon vertex in Type IIB on $AdS_5 \times \mathbf{S}^5$.

The Heisenberg group and the Hilbert space. The operators $\Phi_{RR}(S)$ (and likewise $\Phi_{NS}(T)$) are therefore classified by $H_2(M, \mathbb{Z}_N)$ or equivalently by $K = H^2(M, \mathbb{Z}_N)$. We write these operators now as $\Phi_{RR}(w)$, $\Phi_{NS}(w)$, with $w \in K$. The commutation relation of these operators is still given by 3.6. We can assume that all intersections of S and T occur away from the boundaries, so there is no real modification in the derivation of 3.6 from the canonical commutation relations. Since S and T may have boundaries of the form NC, the intersection number $S \cdot T$ is only

JHEP12(1998)012

well-defined modulo N, but that is good enough to make sense of the phase factor in the commutation relation.

Because of the central factor in the commutation relation 3.6, the group generated by these operators is not just $K \times K = H^2(M, \mathbb{Z}_N) \times H^2(M, \mathbb{Z}_N)$, but is a central extension of $K \times K$ by $\mathbb{Z}_N$, the group of N^{th} roots of unity. We call this central extension W:

$$0 \to \mathbb{Z}_N \to W \to K \times K \to 0\,. \tag{3.8}$$

This central extension is nondegenerate (a statement which, informally, means that the first or second factors of K in the product $K \times K$ give maximal commutative subgroups of W) and is a group of a type known as a finite Heisenberg group.

A "polarization" of such a finite Heisenberg group is a choice of a maximal commutative subgroup of $K \times K$, or more exactly a maximal subgroup that remains commutative when lifted to W. For example, the first or second factor of $K \times K$, or any "diagonal" subgroup obtained from then by an $SL(2, \mathbb{Z})$ transformation, is a polarization. There are also, in general, other polarizations. Picking a polarization of a finite Heisenberg group is a discrete version of picking a representation of the canonical commutation relations $[p, x] = -i\hbar$ for bosons.

One of the main theorems about such finite Heisenberg groups is that, up to isomorphism, such a group has a unique irreducible representation R. This is a discrete version of the fact that the canonical commutation relations for bosons have a unique irreducible representation, up to isomorphism. The representation can be constructed as follows. Since the $\Phi_{RR}(w)$ commute, one can pick a basis of R consisting of their joint eigenstates. Using the nondegeneracy, one can show that there is a vector $|\Omega\rangle \in R$ that is invariant under all of the $\Phi_{RR}(w)$. From irreducibility of R, one can show that $|\Omega\rangle$ is unique (up to multiplication by a scalar) and that R has a basis consisting of the states

$$\Psi_w = \Phi_{NS}(w)|\Omega\rangle, \quad \text{for } w \in H^2(M, \mathbb{Z}_N)\,. \tag{3.9}$$

The action of the group in this basis follows immediately from the commutation relations.

If the quantum Hilbert space $\mathcal{H}'$ of the Chern-Simons theory is an *irreducible* representation of the finite symmetry group W, then it is easy to obtain all the properties promised in section 2.2. For example, we must show that for each polarization of the lattice $\Lambda = \mathbb{Z}^2$ on which $SL(2, \mathbb{Z})$ acts, $\mathcal{H}'$ has a basis Ψ_w, $w \in H^2(M, \mathbb{Z}_N)$. In this context, a polarization is a choice of what we mean by Φ_{RR} and what we mean by Φ_{NS}. The desired basis is constructed in 3.9 for one choice of polarization, and other polarizations are obtained by $SL(2, \mathbb{Z})$ transformations. There is indeed a natural action of $SL(2, \mathbb{Z})$ on $\mathcal{H}'$, since the commutation relation 3.6 by which W is defined is $SL(2, \mathbb{Z})$-invariant. It is not hard to see that the $SL(2, \mathbb{Z})$ action agrees with what was described in section 2.1. The main point is that the $SL(2, \mathbb{Z})$

transformation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.10}$$

which acts by $\tau \to -1/\tau$, corresponds to a change in polarization that maps Φ_{RR} to Φ_{NS}^{-1} and Φ_{NS} to Φ_{RR}. It maps the state $|\Omega\rangle$ invariant under the Φ_{RR}'s to the state

$$|\widetilde{\Omega}\rangle = \frac{1}{N^{\widetilde{b}_2/2}} \sum_{w \in K} \Phi_{NS}(w)|\Omega\rangle\,, \tag{3.11}$$

which is invariant under the Φ_{NS}'s. So it maps $\Psi_v = \Phi_{NS}(v)|\Omega\rangle$ to

$$\begin{aligned} \Phi_{RR}(v)|\widetilde{\Omega}\rangle &= \frac{1}{N^{\widetilde{b}_2/2}} \Phi_{RR}(v) \sum_{w \in K} \Phi_{NS}(w)|\Omega\rangle \\ &= \frac{1}{N^{\widetilde{b}_2/2}} \sum_{w \in K} \exp\left(2\pi i v \cdot w/N\right) \Phi_{NS}(w)|\Omega\rangle\,. \end{aligned} \tag{3.12}$$

This is the discrete Fourier transform familiar from section 2.1. The transformation under $\tau \to \tau + 1$ can be understood similarly.

Except for the fact that we have not shown that the finite group W acts irreducibly in the quantum theory, this really establishes what we wanted. However, we will go on in the rest of this section, for the benefit of curious or intrepid readers, to show what is involved in quantizing the theory directly. In the process, it will become clear that W acts irreducibly on $\mathcal{H}'$.

In the analysis, it will be useful to know the following more general description of the representation R. First of all, if F is any polarization of $K \times K$, then R has a basis that is in one-to-one correspondence with the cosets of $(K \times K)/F$. This is proved by first showing that R has an F-invariant vector $|\Omega\rangle$, and then setting $\Upsilon_\lambda = \Phi_\lambda|\Omega\rangle$, where λ runs over a set of representatives of the cosets of K/F, and for each λ, Φ_λ is the corresponding operator on R. The Υ_λ are the desired basis vectors.

3.3 The torsion-free case; elementary account

Direct quantization of this system is surprisingly subtle, given that it is an abelian free field theory. The reader may in fact wish to jump directly to section 4.

We begin by considering the case that there is no torsion in $H^2(M,\mathbb{Z})$. If $H^2(M,\mathbb{Z})$ is torsion-free, then this group is a lattice Γ. The intersection pairing on $H^2(M,\mathbb{Z})$ gives an integer-valued inner product on Γ; we write the product of $x, y \in \Gamma$ as $x \cdot y$. This inner product is unimodular by Poincaré duality and is even because (as we noted at the outset) M is spin.

The exact sequence of abelian groups $0 \to \mathbb{Z} \xrightarrow{N} \mathbb{Z} \to \mathbb{Z}_N \to 0$ leads to a long exact sequence of cohomology groups which reads in part

$$\ldots H^2(M,\mathbb{Z}) \xrightarrow{N} H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{Z}_N) \to H^3(M,\mathbb{Z}) \xrightarrow{N} H^3(M,\mathbb{Z}) \ldots \tag{3.13}$$

JHEP12(1998)012

If the cohomology of M is torsion-free, then the map $H^3(M,\mathbb{Z})\xrightarrow{N}H^3(M,\mathbb{Z})$ is injective. Hence exactness of 3.13 says in the torsion-free case that

$$H^2(M,\mathbb{Z}_N) = H^2(M,\mathbb{Z})/NH^2(M,\mathbb{Z}) = \Gamma/N\Gamma\,. \tag{3.14}$$

In other words, in the absence of torsion, a $\mathbb{Z}_N$ class is the mod N reduction of an integral class.

In the torsion-free case, the world-sheet theta angles α_{NS} and α_{RR} are a complete set of gauge-invariant functions on the classical phase space. They are canonically conjugate variables, a fact that we have already used in the analysis of the symmetries in section 3.2. A precise form of this statement is that if we pick for the lattice Γ a basis in which the inner product is g_{ij}, $1 \le i,j \le b_2$, and write α^i_{NS}, α^i_{RR} for the corresponding components of α_{NS} and α_{RR}, then the Poisson brackets are

$$\{\alpha^i_{RR},\alpha^j_{NS}\} = \frac{2\pi}{N}g^{ij}\,. \tag{3.15}$$

According to the description at the end of section 2.2, we are to pick a polarization, quantize, and determine the resulting Hilbert space $\mathcal{H}'$. We pick a polarization[5] by regarding α_{NS} as the "position" variable and

$$\beta_{RR} = \frac{N}{2\pi}\alpha_{RR} \tag{3.16}$$

as the conjugate momentum. Note that, since the periods of α_{RR} are defined mod 2π, then those of β_{RR} are defined mod N, so, classically, β_{RR} is an element of $H^2(M,\mathbf{R})/NH^2(M,\mathbb{Z}) = V/N\Gamma$, where $V = H^2(M,\mathbf{R})$ and $\Gamma = H^2(M,\mathbb{Z})$ is regarded as a lattice in V.

Quantum mechanically there is a very severe additional constraint on β_{RR}. Since the "position" variable α_{NS} is a periodic variable that takes values in $H^2(M,\mathbb{Z})/2\pi\Gamma$, the conjugate momentum β_{RR} takes values in the lattice Γ^* dual to Γ. But Poincaré duality tells us that $\Gamma^* = \Gamma$, so β_{RR} takes values in Γ. Since β_{RR} is in addition defined modulo $N\Gamma$, it takes values in $\Gamma/N\Gamma$. According to 3.14, this is the same as $H^2(M,\mathbb{Z}_N)$.

So we get the desired result. There is one quantum state – one momentum state Ψ_w – for every momentum vector $w \in \Gamma/N\Gamma = H^2(M,\mathbb{Z}_N)$.

Moreover, the $SL(2,\mathbb{Z})$ transformation $\tau \to -1/\tau$ exchanges the position α_{NS} with the momentum α_{RR}. Hence it acts as a Fourier transform, in agreement with the gauge theory result 2.3.

3.4 More rigorous approach

The discussion in section3.3 was actually somewhat informal, and for the interested reader we will now give some hints for a more precise treatment.

[5] For some background on quantization of Chern-Simons theories, see [19, 20].

JHEP12(1998)012

Let T be the torus $T = V/\Gamma$ with $V = H^2(M, \mathbf{R})$ and $\Gamma = H^2(M, \mathbb{Z})$. The phase space of the system is $T' = T \times T$. The first step in quantizing the theory is to find the appropriate line bundle $\mathcal{M}$ over the phase space. $\mathcal{M}$ should be endowed with a connection whose curvature equals the symplectic two-form of the theory. Once the right line bundle is found, quantization is carried out by picking a complex structure and taking holomorphic sections of $\mathcal{M}$, or by using a real polarization and making a more precise version of the informal discussion of section 3.3, or by finding and using a finite Heisenberg group as in section 3.2. But any approach to quantization involves finding the line bundle at the outset.

Since N appears in the symplectic form as a multiplicative factor, $\mathcal{M}$ will be of the form $\mathcal{L}^N$, where $\mathcal{L}$ is the line bundle that we would get at $N = 1$. (The relation $\mathcal{M} = \mathcal{L}^N$ will be obvious from the Chern-Simons construction of the line bundle that we give presently.)

The construction of the line bundle is most naturally made directly from the Chern-Simons action [21, 22]. Let p be a point in the classical phase space T' given by a pair $(B_{RR}, B_{NS}) = (a, b)$. Let 0 be the origin in T' (the point with $B_{RR} = B_{NS} = 0$). Then we describe the fiber $\mathcal{L}_p$ of $\mathcal{L}$ at p as follows. For any path γ in T' from 0 to p, $\mathcal{L}_p$ has a basis vector ψ_γ, of norm 1. If γ and γ' are two such paths, we declare that

$$\psi_{\gamma'} = e^{iL_{CS}} \psi_\gamma \,, \tag{3.17}$$

where the Chern-Simons Lagrangian L_{CS} is understood in the following sense. The path γ from 0 to p determines a "time"-dependent pair of B-fields $(B_{RR}(t), B_{NS}(t))$, where $(B_{RR}(0), B_{NS}(0)) = (0, 0)$ and $(B_{RR}(1), B_{NS}(1)) = (a, b)$. We can fit these together to make a pair (B_{RR}, B_{NS}) over $M \times I$, where I is a unit interval. Likewise, γ' determines a pair of B-fields over $M \times I$. By gluing γ' to γ, with opposite orientation for γ', we get a B-field pair over the closed five-manifold $M \times \mathbf{S}^1$. L_{CS} in 3.17 is the Chern-Simons action evaluated for this field.

It can be shown that the line bundle $\mathcal{L}$, whose fiber at each $p \in T'$ was just described, is endowed with a natural connection, whose curvature is the symplectic form. The monodromy M_C of this line bundle around any circle $C \subset T'$ is as follows. The circle $C \subset T'$ determines a B-field pair over $M \times C$, and the monodromy is

$$Q(C) = \exp(-L_{CS}) \,, \tag{3.18}$$

with L_{CS} the Chern-Simons action of this pair.

Having found the line bundle, we are ready to quantize. Quantum states are suitable sections of $\mathcal{M} = \mathcal{L}^N$, with the details depending on a choice of polarization.

In the present case, we can make the description of $\mathcal{L}$ much more explicit. In general, a line bundle with connection over any manifold Z is determined up to isomorphism by giving its monodromies around arbitrary loops in Z. Once the curvature is specified, it suffices to consider a set of loops generating the fundamental

JHEP12(1998)012

group of Z. In the present instance, Z is the torus $T' = T \times T = (V \times V)/(\Gamma \times \Gamma)$. The fundamental group of T' is generated by straight lines from the origin in $V \times V$ to lattice points in $\Gamma \times \Gamma$. Let (x, y) be such a lattice point. A straight line from the origin to this lattice point is given by the family of B-field pairs

$$\begin{aligned} B_{RR} &= 2\pi t x\,, \\ B_{NS} &= 2\pi t y\,, \end{aligned} \tag{3.19}$$

with $0 \leq t \leq 1$. At $t = 1$, $(B_{RR}, B_{NS}) = (2\pi x, 2\pi y)$ are gauge-equivalent to 0. So the family $(B_{RR}(t), B_{NS}(t))$ gives a closed loop in the phase space. The monodromy around this loop can be computed by direct evaluation of $L_{CS} = -(i/2\pi)\int_{M\times \mathbf{S}^1} B_{RR}\wedge dB_{NS}$,[6] and is

$$Q(x, y) = (-1)^{x\cdot y}\,. \tag{3.20}$$

The relation

$$Q(x + x', y + y') = Q(x, y)Q(x', y')(-1)^{x\cdot y' + y\cdot x'} \tag{3.21}$$

follows from this, and signals, as explained in connection with eqn. (2.17) of [8], that the line bundle $\mathcal{L}$ has the desired curvature and first Chern class. The formula for $Q(x, y)$ is clearly invariant under all of the lattice symmetries of Γ, and thus under all diffeomorphisms of M. Let us check that it is also invariant under $SL(2, \mathbb{Z})$, acting in the standard fashion

$$\begin{aligned} x &\to ax + by\,, \\ y &\to cx + dy\,, \end{aligned} \tag{3.22}$$

with a, b, c, d integers such that $ad - bc = 1$. A small computation shows that $Q(x, y)$ is invariant under this transformation for all such a, b, c, and d if and only if x^2 and y^2 are both even. But the manifold M is spin, and the intersection form on $H^2(M, \mathbb{Z})$, for M a four-dimensional spin manifold, is always even. So x^2 and y^2 are even, and Q has the expected symmetries. Since Q determines the line bundle $\mathcal{L}$, $\mathcal{L}$ has these symmetries also.

At this point, the reader may wonder precisely why the spin condition on M was needed. The line bundle $\mathcal{L}$ was determined directly from the Chern-Simons action, which appears to be completely $SL(2, \mathbb{Z})$-invariant without assuming that M is spin; so why did the spin structure enter? The answer will make it clear that up to the present point, we have indulged in a small sleight of hand. There is actually a subtlety in defining the Chern-Simons action for a pair of B-fields (B_{RR}, B_{NS}) on a

[6] To put this "direct evaluation" on a rigorous basis is slightly subtle as the B-fields on $M \times \mathbf{S}^1$ are topologically nontrivial. The basic idea of a rigorous evaluation in this situation is contained in Example 1.16 in [9].

JHEP12(1998)012

five-manifold X, even if X has no boundary. If X is the boundary of a six-manifold Y, and the B-fields extend over Y, then the action is readily defined as

$$I_{CS} = -\frac{i}{2\pi}\int_Y H_{RR}\wedge H_{NS}\,. \tag{3.23}$$

This formula is completely $SL(2,\mathbb{Z})$-invariant, proving that $SL(2,\mathbb{Z})$ is a symmetry if Y exists. If no such Y exists, then as mentioned in the introduction, one must define the action by $\exp(-I_{CS}) = B_{RR} * B_{NS}$, with $*$ the multiplication in Cheeger-Simons cohomology. From Theorem 1.11 of [9], one can deduce that $I_{CS}(B_{RR}, B_{NS}) = I_{CS}(B_{NS}, -B_{RR})$ and that $I_{CS}(B_{RR}, B_{NS}) = I_{CS}(B_{RR} + 2B_{NS}, B_{NS})$. Moreover $SL(2,\mathbb{Z})$ holds if and only if one has the more precise property $I_{CS}(B_{RR}, B_{NS}) = I_{CS}(B_{RR}+B_{NS}, B_{NS})$, but this does not follow from the theorem and is evidently not true in general if X is not spin. The $SL(2,\mathbb{Z})$ invariance of the explicit formula 3.20 that determines the line bundle shows that, at least in canonical quantization, there is no such difficulty in the spin case.

Quantization. Having thus defined the line bundle, we now wish to carry out quantization. This may be done by picking a polarization and quantizing, but we prefer to make contact with the discussion of section 3.2 by exhibiting the discrete Heisenberg group.

Since the phase space T' is a torus, some obvious symmetries of the phase space (as a symplectic manifold) are translations of the torus, say $(B_{RR}, B_{NS}) \to (B_{RR}, B_{NS}) + 2\pi(a,b)$, with $a, b \in V/\Gamma$. However, such translations generally do not leave fixed the line bundle $\mathcal{L}$. Under the indicated translation, the monodromy $Q(x,y)$ computed above transforms by

$$Q(x,y) \to \exp(2\pi i(a\cdot y - b\cdot x))Q(x,y)\,. \tag{3.24}$$

Thus, the monodromies are invariant, for all x, y, only if $a, b \in \Gamma$, in which case the translation by $2\pi(a,b)$ acts trivially on the torus T'.

So if the quantum line bundle is $\mathcal{L}$, there are no nontrivial translation symmetries. We are more generally interested in the "level N" theory in which the quantum line bundle is $\mathcal{M} = \mathcal{L}^N$. In this case, the monodromies are Q^N, and it is sufficient for Q^N to be invariant. For this, it is enough that Na and Nb should be lattice points. So the translation by $2\pi(a,b)$ is a symmetry of the quantum theory whenever a and b are both of order N. The points of order N are classified by $\frac{1}{N}\Gamma/\Gamma = \Gamma/N\Gamma = H^2(M,\mathbb{Z}_N)$.

Let us try to define as precisely as possible the symmetry $T_{a,b}$ of translation by $2\pi(a,b)$. Such a symmetry exists because the pullback of $\mathcal{M}$ by the translation in question is isomorphic to $\mathcal{M}$. The operator $T_{a,b}$ is uniquely determined up to multiplication by a complex scalar of modulus 1. That scalar can be restricted by requiring (since the translation by $2\pi(Na, Nb)$ is trivial) that $T_{a,b}^N = 1$. This leaves

JHEP12(1998)012

an ambiguity consisting of multiplication by N^{th} roots of unity. There is no natural way to fix that remaining ambiguity, so we make any arbitrary choice.

One might think that the translation operators $T_{a,b}$ would commute. That is not so, because in the presence of a magnetic field (the symplectic form on T' serves as the "magnetic field") translations do not commute. Including in the standard way the effects of the magnetic field, the actual relation is

$$T_{a,b}T_{a',b'} = T_{a',b'}T_{a,b}\exp\left(2\pi i(a\cdot b' - b\cdot a')/N\right). \tag{3.25}$$

This is the Heisenberg group found in section 3.2. The quantum theory can now be analyzed as in section 3.2, with the important difference that we can now show that W acts irreducibly. For instance, this follows by using a complex polarization and Riemann-Roch theorem to determine the dimension of the quantum Hilbert space, which coincides with that of the irreducible representation R of W. For that matter, the informal treatment in section 3.3 was precise enough, at least for large enough N, to count the quantum states within a positive integer factor, and thus to deduce that W acts irreducibly.

3.5 Restriction on $[H]$

Now we want to begin to consider what happens if there is torsion in the cohomology of M.

We first note that B-fields on M are characterized topologically by a characteristic class $[H]\in H^3(M,\mathbb{Z})$. The Gauss law constraint implies that $H=dB$ is zero as a differential form, so $[H]$ must be torsion. At first sight, it appears that there is no other restriction, and that in quantizing the Chern-Simons theory, $[H_{RR}]$ and $[H_{NS}]$ can be arbitrary torsion elements of $H^3(M,\mathbb{Z})$.

Actually, there is a very important further restriction. This is that if the Chern-Simons theory is taken at level N, then $[H_{RR}]$ and $[H_{NS}]$ must be N-torsion, that is $N[H_{RR}]=N[H_{NS}]=0$.

To obtain this result, we will consider the partition function of the theory on the five-manifold $X=M\times\mathbf{S}^1$. In topological field theory, this partition function equals the dimension of the space of physical states on M, and so in particular every physical state contributes.

We consider classical configurations on $M\times\mathbf{S}^1$ in which, when restricted to M (that is to $M\times P$ for any $P\in\mathbf{S}^1$), $[H_{NS}]$ has some given value. We want to show that the contribution to the path integral of such configurations is zero unless $N[H_{NS}]=0$. The same argument, with B_{RR} and B_{NS} exchanged, shows that the contribution is zero unless also $N[H_{RR}]=0$ when restricted to M.

The vanishing will come upon adding to B_{RR} a flat B-field B' that is topologically trivial if restricted to M, but nontrivial on $M\times\mathbf{S}^1$. (We restrict B' to be trivial on M because we consider a path integral in which $[H_{RR}]$ and $[H_{NS}]$ are both specified

JHEP12(1998)012

on M; we want to show that summing over what happens in the "time" direction gives the desired restriction on the initial data.) Under $B_{RR} \to B_{RR} + B'$, the path integrand $e^{-L_{CS}}$ of Chern-Simons theory transforms by

$$\exp(-L_{CS}) \to \exp(-L_{CS}) \exp\left(\frac{iN}{2\pi} \int_{M \times \mathbf{S}^1} B' \wedge dB_{NS}\right), \tag{3.26}$$

where the exponential factor needs some clarification (which will be given shortly) because the B-fields in question are topologically nontrivial, but is hopefully clear. If the exponential factor in 3.26 does not equal 1, then the path integral will vanish after summing over B'.

We must therefore understand what is meant by the factor that we write symbolically as

$$\exp\left(\frac{i}{2\pi} \int_X B' \wedge dB_{NS}\right), \tag{3.27}$$

if B' and B are flat but topologically nontrivial. This is determined in eqn. 1.14 of [9]; the intuitively natural result can be explained as follows. We can classify flat B-fields in five dimensions either by $H^2(X, U(1))$ or by the characteristic class $[H] \in H^3(X, \mathbb{Z})$. (The first classification describes the B-field up to gauge transformation, and the second only describes its topological type.) There is a natural, nondegenerate pairing

$$H^2(X, U(1)) \times H^3(X, \mathbb{Z}) \to H^5(X, U(1)) = U(1) \tag{3.28}$$

given by Poincaré and Pontryagin duality. For $B' \in H^2(X, U(1))$ and B_{NS} regarded as an element of $H^3(X, \mathbb{Z})$, we write this pairing as $E(B', B_{NS})$. Cheeger and Simons show that in this situation $B' * B_{NS} = E(B', B_{NS})$, so we must interpret the phase factor in 3.27 as $E(B', B_{NS})$.

Because of the factor of N in the exponent 3.26, the factor that must be 1 for B_{NS} to contribute to the path integral is actually $E(B', B_{NS})^N = E(B', NB_{NS})$. Nondegeneracy of the pairing 3.28 means that for this to equal 1 for all B', we need $NB_{NS} = 0$, up to gauge transformation. In other words, we need $N[H_{NS}] = 0$. In our case with $X = M \times \mathbf{S}^1$, since we assume $B' = 0$ when restricted to M, the constraint is that $N[H_{NS}] = 0$ when restricted to M. The is the restriction that we aimed to prove.

The restriction that we have found should be viewed as a quantum extension of Gauss's law, to incorporate torsion. Gauss's law enters in path integrals as a constraint that states must obey, in order to propagate in time. Such propagation is precisely what we have been analyzing. The classical Gauss law says that $H_{RR} = H_{NS} = 0$ as a differential form; the quantum Gauss law says that in addition $N[H_{RR}] = N[H_{NS}] = 0$.

3.6 Quantization in the presence of torsion

We will now analyze the quantum Hilbert space $\mathcal{H}'$ of the Chern-Simons theory, allowing for the possible existence of torsion in $H^2(M, \mathbb{Z}_N)$. In fact, to clarify things,

JHEP12(1998)012

we will consider first the case that the cohomology of M contains only torsion – the opposite case from what we considered in sections 3.3 and 3.4.

After imposing Gauss's law, the physical data consists of a pair of flat B-fields B_{RR}, B_{NS}. From what we have seen in section 3.5, we should impose a discrete form of Gauss's law stating that $N[H_{RR}] = N[H_{NS}] = 0$. Let L_N be the subgroup of $H^3(M, \mathbb{Z})$ consisting of points of order N. The two B-fields determine a pair of points in L_N. If the cohomology of M is pure torsion, there are no additional data to quantize. Each pair $(x, y) \in L_N \times L_N$ contributes one quantum state $\Psi_{x,y}$, and the $\Psi_{x,y}$ give a basis of the physical Hilbert space $\mathcal{H}'$.

We want to compare this to the description from section 2: for each choice of polarization, $\mathcal{H}'$ should have a basis Ψ_w, for $w \in K = H^2(M, \mathbb{Z}_N)$. In an equivalent version presented in section 3.2, $\mathcal{H}'$ should furnish an irreducible description of the finite Heisenberg group W:

$$0 \to \mathbb{Z}_N \to W \to K \times K \to 0\,. \tag{3.29}$$

For this, we must understand the structure of $H^2(M, \mathbb{Z}_N)$ when there is torsion. From the exact sequence of abelian groups $0 \to \mathbb{Z} \xrightarrow{N} \mathbb{Z} \to \mathbb{Z}_N \to 0$, with the first map being multiplication by N and the second reduction modulo N, we get a long exact sequence

$$\ldots \to H^2(M, \mathbb{Z}) \xrightarrow{N} H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}_N) \to H^3(M, \mathbb{Z}) \xrightarrow{N} H^3(M, \mathbb{Z}) \to \ldots \tag{3.30}$$

This gives a short exact sequence

$$0 \to E_N \to H^2(M, \mathbb{Z}_N) \to L_N \to 0, \tag{3.31}$$

with $E_N = H^2(M, \mathbb{Z})/NH^2(M, \mathbb{Z})$ the subgroup of $H^2(M, \mathbb{Z}_N)$ consisting of classes that are the reduction of an integer class.

We recall that there is a nondegenerate pairing $H^2(M, \mathbb{Z}_N) \times H^2(M, \mathbb{Z}_N) \to \mathbb{Z}_N$ given by Poincaré duality; it was used in section 3.2 to construct the finite Heisenberg group. If the cohomology of M is purely torsion, then E_N consists entirely of classes that are reductions of torsion classes. In this case, the pairing on $H^2(M, \mathbb{Z}_N) \times H^2(M, \mathbb{Z}_N)$ is zero when restricted to $E_N \times E_N$ (since in integer cohomology the cup product vanishes for torsion classes), and the self-duality of $H^2(M, \mathbb{Z}_N)$ reduces to a duality between E_N and L_N.

The Heisenberg group W in this situation has a maximal commutative subgroup that comes from the subgroup $E_N \times E_N$ of $H^2(M, \mathbb{Z}_N) \times H^2(M, \mathbb{Z}_N)$. According to the remark at the end of section 3.2, the representation R has therefore a basis in 1-1 correspondence with the cosets in $(K \times K)/(E_N \times E_N)$. That quotient is isomorphic to $L_N \times L_N$, so we get one basis vector $\Psi_{x,y}$ for each pair $(x, y) \in L_N \times L_N$. This is the description we found for the physical Hilbert space $\mathcal{H}'$, so we confirm the expected isomorphism of R with $\mathcal{H}'$.

JHEP12(1998)012

In particular, we see again that $\mathcal{H}'$ is an irreducible representation of the discrete Heisenberg group.

The general case. The general case in which the cohomology of M contains torsion but is not purely torsion is a mixture of the cases that we have already considered.

In this case, the cohomology classes of $x = [H_{RR}]$, $y = [H_{NS}]$ determine elements $x, y \in L_N$ which are part of the data. In addition, one must quantize the α's. We can pick a polarization in which the quantum states are regarded as functions of α_{NS} as well as x, y.

As we found in section 3.3, quantization of the α's gives one state for every topologically trivial flat B_{NS}-field of order N. Including also the dependence on y means that we drop the requirement that B_{NS} should be topologically trivial; we get one state for every flat B_{NS} field of order N whether it is topologically trivial or not. We let M_N be the group of flat B-fields of order N. Including also the choice of x (which is a point in L_N), we see that the quantum Hilbert space $\mathcal{H}'$ has a basis consisting of one vector for every element of $L_N \times M_N$. This strongly suggests that there might be a polarization of the discrete Heisenberg group determined by a subgroup F of $K \times K$ such that $(K \times K)/F = L_N \times M_N$. In that case, the irreducible representation R of the discrete Heisenberg group can be identified with $\mathcal{H}'$ as desired.

In fact, we can take $F = E_N \times T_N$, where T_N is the subgroup of $H^2(M, \mathbb{Z}_N)$ consisting of elements that are the reduction mod N of torsion classes in $H^2(M, \mathbb{Z})$. F is a commutative subgroup (even when lifted to W) because the pairing of two elements in $H^2(M, \mathbb{Z})$, one of which is torsion, is zero. Since $(K \times K)/F = (K \times K)/(E_N \times T_N) = (K/E_N) \times (K/T_N)$, the claim that $(K \times K)/F = L_N \times M_N$ is equivalent to the existence of two exact sequences:

$$\begin{aligned} &0 \to E_N \to H^2(M, \mathbb{Z}_N) \to L_N \to 0\,, \\ &0 \to T_N \to H^2(M, \mathbb{Z}_N) \to M_N \to 0\,. \end{aligned} \tag{3.32}$$

The first is in 3.31, and the second can be derived from the sequence

$$0 \to \mathbb{Z}_N \to U(1) \xrightarrow{N} U(1) \to 0 \tag{3.33}$$

(the first map is the inclusion of the points of order N in $U(1)$, and the second is $a \to a^N$). From this sequence we get a long exact cohomology sequence

$$0 \to H^1(M, U(1))/NH^1(M, U(1)) \to H^2(M, \mathbb{Z}_N) \to H^2(M, U(1))_N \to 0\,. \tag{3.34}$$

Here $H^2(M, U(1))_N$ is the kernel of $H^2(M, U(1)) \xrightarrow{N} H^2(M, U(1))$; this is the group of flat B-fields of order N, or in other words is M_N. Also, $H^1(M, U(1))$ classifies flat $U(1)$ bundles. Such a bundle has a first Chern class which is a torsion element of $H^2(M, \mathbb{Z})$; and the map from $H^1(M, U(1))/NH^1(M, U(1))$ to $H^2(M, \mathbb{Z}_N)$

JHEP12(1998)012

is given by mod N reduction of the first Chern class. Conversely, every torsion element of $H^2(M,Z)$ is the first Chern class of a flat line bundle. So the image of $H^1(M,U(1))/NH^1(M,U(1))$ in $H^2(M,\mathbb{Z}_N)$ consists precisely of the classes that are mod N reductions of torsion classes, or in other words this image is T_N. This completes the explanation of the second exact sequence in 3.32.

4. The $(0,2)$ theory in six dimensions

In this section, we will consider the analogous issues for the $(0,2)$ conformal field theory in six-dimensions.[7] We consider only the A_N theory, that is, the theory associated with M-theory on $AdS_7 \times \mathbf{S}^4$, with N units of four-form flux on $\mathbf{S}^4$. We will, in essence, be determining the analog for the A_N case of the (0,2) theory of the space of conformal blocks in two-dimensional rational conformal field theory. A difference from the problem considered in sections 2 and 3 is that there is no obvious candidate from classical geometry for what the answer should be.

To study the $(0,2)$ theory on a six-dimensional spin-manifold M, we consider M-theory on spacetimes that look near infinity like $X \times \mathbf{S}^4$, where X has M for conformal boundary. By analogy with gauge theory in four-dimensions, we suspect that for general M, the partition function will not be a number but a vector in some finite-dimensional Hilbert space associated with M.

A mechanism analogous to what we studied in sections 2 and 3 immediately presents itself. The long wavelength theory on X has a three-form field C, whose effective action contains the Chern-Simons term

$$L_{CS} = -i\frac{1}{2}\frac{N}{2\pi}\int_X C \wedge dC\,. \tag{4.1}$$

If N is odd, then because of the $1/2$ in 4.1, to make sense of this theory requires a spin structure on X. (This is explained in [8], where the case $N=1$ was considered and the factor of $1/2$ in 4.1, which arises from a similar factor of $1/6$ in M-theory, played an important role. Some additional important details, such as a gravitational correction to 4.1, were also described there.) There is no harm in this, because in any event, M-theory on $X \times \mathbf{S}^4$ requires a spin structure on X. If N is even or a spin structure is picked on X, then quantization is carried out rather as in section 3.2 by introducing a Heisenberg group associated with $H^3(M,\mathbb{Z}_N)$. This Heisenberg group contains an operator Φ_v for each $v \in H^3(M,\mathbb{Z}_N)$, with[8]

$$\Phi_v\Phi_w = \Phi_w\Phi_v \exp\left(2\pi i v\cdot w/N\right). \tag{4.2}$$

[7] Some related issues involving a discrete flux in the $(0,2)$ model have been discussed in [23].

[8] From one point of view, the need for a spin structure arises because the following formula does not quite determine the Heisenberg group. To have a Heisenberg group, we need a multiplication law $\Phi_v\Phi_w = c(v,w)\Phi_{v+w}$, with $c(v,w)c(w,v)^{-1} = \exp(2\pi i v\cdot w/N)$. Since $v\cdot w$ is only well-defined modulo N, there is no elementary formula for $c(v,w)$ except to take an ambiguous square root $c(v,w) = \pm\sqrt{\exp(2\pi i v\cdot w/N)}$. A spin structure determines a consistent set of square roots. In

JHEP12(1998)012

We thus have a group extension

$$0 \to \mathbb{Z}_N \to W \to H^3(M, \mathbb{Z}_N) \to 0\,, \tag{4.3}$$

and the quantum Hilbert space $\mathcal{H}$ is an irreducible representation of W. (Irreducibility can be proved by a detailed analysis along the lines of sections 3.3- 3.4.)

One important difference from the four-dimensional case is that W is defined using only one copy of $H^3(M, \mathbb{Z}_N)$ (rather than two copies of $H^2(M, \mathbb{Z}_N)$ as in four dimensions). So there is no way to pick a polarization and give an explicit description of $\mathcal{H}$ without using some detailed properties of W and breaking the symmetries of the finite group $H^3(M, \mathbb{Z}_N)$. One simple case is $M = \mathbf{S}^1 \times Y$, with Y a five-dimensional spin manifold. In this case, $H^3(M, \mathbb{Z}_N) = H^2(Y, \mathbb{Z}_N) \oplus H^3(Y, \mathbb{Z}_N)$. A polarization of $H^3(M, \mathbb{Z}_N)$ is given by its subgroup $H^3(Y, \mathbb{Z}_N)$. The quotient is $H^3(M, \mathbb{Z}_N)/H^3(Y, \mathbb{Z}_N) = H^2(Y, \mathbb{Z}_N)$. Hence, the Hilbert space $\mathcal{H}$ has a basis with a vector Ψ_w for each $w \in H^2(Y, \mathbb{Z}_N)$. This result has a simple explanation. The $(0, 2)$ conformal field theory, if compactified on $\mathbf{S}^1$, gives a theory that looks at low energies like $SU(N)/\mathbb{Z}_N$ gauge theory in five dimensions. If this is further compactified on the five-manifold Y, the gauge theory has a discrete magnetic flux taking values in $H^2(Y, \mathbb{Z}_N)$, and this leads to the Hilbert space that we found.

5. 't Hooft and Wilson lines in four dimensions

In this concluding section, we will analyze the behavior of the $\mathcal{N} = 4$ super Yang-Mills theory on a four-manifold M with 't Hooft and Wilson loops included.

We begin on a five-manifold X by supplementing the Chern-Simons Lagrangian L_{CS} by couplings of the B-fields to string worldsheets. The strings in question are fundamental strings coupling to B_{NS} and D-strings coupling to B_{RR}. We take the D-string and elementary string worldsheets to be closed surfaces S_{RR} and S_{NS}, which are not necessarily connected. Including the couplings to the strings, the Lagrangian is

$$\widehat{L} = N I_{CS}(B_{RR}, B_{NS}) - \frac{i}{2\pi} \int_{S_{RR}} B_{RR} - \frac{i}{2\pi} \int_{S_{NS}} B_{NS}\,. \tag{5.1}$$

If X has conformal boundary M, then the boundary of S_{RR} and S_{NS} should be [24, 25] one-manifolds C_{RR} and C_{NS} in M on which 't Hooft and Wilson loops are inserted. (The C's are not necessarily connected.)

---

fact, having a Heisenberg group extension and not just a commutation relation is equivalent — roughly as we saw in section 3.3 — to having a suitable line bundle over the classical phase space. As analyzed in detail in [8], defining this line bundle depends on having a spin structure on M. The Chern-Simons theory that we have studied in this paper in five dimensions likewise needs a spin structure, as we saw in section 3.3.

JHEP12(1998)012

The path integral on X will take values in a Hilbert space associated with the boundary. Near the boundary, we have $X = M \times \mathbf{R}$, where we view $\mathbf{R}$ as the "time" direction. Near the boundary, we can take $S_{RR} = C_{RR} \times \mathbf{R}$ and $S_{NS} = C_{NS} \times \mathbf{R}$. Quantization is carried out formally as in section 3.1. The Gauss law constraint contains an extra contribution from the strings and reads

$$\begin{aligned} NH_{NS} &+\delta(C_{RR}) = 0\,, \\ -NH_{RR} &+\delta(C_{NS}) = 0\,. \end{aligned} \tag{5.2}$$

Here, for $C = C_{RR}$ or C_{NS}, $\delta(C)$ is a delta-function supported on C, representing a cohomology class in $H^3(M,\mathbb{Z})$ that is dual to C. We will call this class $[C]$.

If the equations 5.2 have solutions, then as these equations are linear, by shifting the B-fields by any solution, we can reduce to the previous case in which the phase space that must be quantized is governed by $H_{RR} = H_{NS} = 0$. Hence, the quantum Hilbert space is isomorphic to the one obtained without the Wilson lines, though not canonically so as some arbitrary choice of a solution of Gauss's law has to be made in identifying the phase spaces with and without the Wilson lines. However, the condition that equations 5.2 should have any solutions at all is nontrivial. The requirement that these equations have a solution for some B-fields is simply that the elements $[C_{RR}]$, $[C_{NS}]$ should be divisible by N in $H^3(M,\mathbb{Z})$. (When torsion is present, a full justification of this statement requires a torsion extension of the Gauss's law constraint, along lines presented in section 3.5.)

For example, if $M = Y \times \mathbf{S}^1$, with Y a three-manifold, then the requirement is that the C's should wrap around $\mathbf{S}^1$ a number of times divisible by N. This statement has a simple intuitive interpretation. It means that the Wilson line wrapped on C_{NS} (or the magnetic counterpart wrapped on C_{RR}) is invariant under the thermal symmetry group $F = H^1(M,\mathbb{Z}_N)$. This way of seeing the thermal symmetry group is actually a close cousin of the argument in [7].

For the rest of this section, we specialize to the case $M = \mathbf{S}^4$. In this case, since $H^3(M,\mathbb{Z}) = 0$, there is no topological obstruction to solving the Gauss law constraints, regardless of what the C's might be. Moreover, the space of solutions mod gauge transformations is a single point (since the cohomology groups that classify flat B-fields are all zero). So the quantum Hilbert space is one-dimensional.

The expectation value of an arbitrary product of Wilson and 't Hooft loops on $\mathbf{S}^4$ thus takes values in a one-dimensional Hilbert space $\mathcal{H}$. However, the expectation value cannot in general be understood as an ordinary complex number, because there is no natural way to pick a basis vector in $\mathcal{H}$ and identify $\mathcal{H}$ with the complex numbers $\mathbf{C}$. Let us explore the matter somewhat more thoroughly and see what is involved in picking a basis vector.

The presence of the Wilson loop on C_{NS} leads to the presence in the path integral

JHEP12(1998)012

on $X \times \mathbf{S}^5$ of a factor

$$\exp\left(i\int_{S_{NS}} B_{NS}\right). \tag{5.3}$$

This factor is well-defined as a complex number if S_{NS} is a closed surface. If not, the factor takes values in a one-dimensional Hilbert space $\mathcal{H}_{NS}$. $\mathcal{H}$ is the tensor product of $\mathcal{H}_{NS}$ with a similar one-dimensional Hilbert space $\mathcal{H}_{RR}$ associated with the D-strings, as well as a factor independent of all 't Hooft and Wilson loops that enters in defining the bulk term in the action. To trivialize $\mathcal{H}_{NS}$, we should give a surface $D_{NS} \subset M$ of boundary $-C_{NS}$ (the minus sign is a reversal of orientation), and replace 5.3 by

$$\exp\left(i\int_{S_{NS}+D_{NS}} B_{NS}\right). \tag{5.4}$$

Here $S_{NS} + D_{NS}$ is a closed surface in X, so 5.4 is well-defined as a number.[9] We must, however, investigate the dependence on the choice of D_{NS}. Let D'_{NS} be another choice. Then in replacing D_{NS} by D'_{NS}, 5.4 is multipled by

$$\Psi = \exp\left(i\int_E B_{NS}\right), \tag{5.5}$$

where E is the closed surface in M defined by $E = D'_{NS} - D_{NS}$. Since E is contained in $\mathbf{S}^4$, whose homology vanishes, it is the boundary of a three-manifold $Y \subset M$, and we can write

$$\Psi = \exp\left(i\int_Y H_{NS}\right). \tag{5.6}$$

If there are no D-strings, then H_{NS} vanishes by the equations of motion, and the factor Ψ in the path integral can be eliminated by a redefinition of the fields, in fact, by $B_{RR} \to B_{RR} + (2\pi/N)\delta(Y)$. ($Y$ is of codimension two in X, and $\delta(Y)$ is a two-form dual to Y.) But if there are D-strings, this is not quite a symmetry. Under this transformation, the D-string factor $\exp(i\int_{S_{RR}} B_{RR})$ in the path integral picks up a phase $\exp\left((2\pi i/N)Y \cdot C_{RR}\right)$, where $Y \cdot C_{RR}$ denotes the oriented intersection number of Y and C_{RR}. That intersection number is (by definition) the linking number $\ell(E, C_{RR})$. Hence, under a change in trivialization of $\mathcal{H}_{NS}$ coming from a change in D_{NS}, the path integral is multiplied by

$$\exp(2\pi i\ell(E, C_{RR})/N). \tag{5.7}$$

This factor is, of course, 1 if there are no 't Hooft loops ($C_{RR} = 0$). So the expectation value of a product of Wilson loops only (or likewise of 't Hooft loops only) can be viewed as an ordinary number. But the expectation value of a product consisting of both Wilson and 't Hooft loops, though well-defined as a vector in the one-dimensional Hilbert space $\mathcal{H}$, is ambiguous up to multiplication by an N^{th} root of unity if one wishes to interpret it as an ordinary complex number.

[9] D_{NS} should lie in M, not just in X, so as to trivialize $\mathcal{H}$ (which only depends on M) once and for all, independent of the choice of X.

JHEP12(1998)012

An ambiguity of just this nature can be seen by standard gauge theoretic methods. In that context, one begins with the problem of what it means in $SU(N)/\mathbb{Z}_N$ gauge theory to define the expectation value of a Wilson loop, wrapped on a circle C_{NS}, in the fundamental representation of $SU(N)$. For this, we would want a lifting of the $SU(N)/\mathbb{Z}_N$ bundle to $SU(N)$ at least along C_{NS}. It suffices to give a surface D_{NS} with boundary C_{NS}. For, as $H^2(D_{NS}, \mathbb{Z}_N) = 0$, the $SU(N)/\mathbb{Z}_N$ bundle when restricted to D_{NS} can be lifted to $SU(N)$, and though the lifting is not unique, the nonuniqueness does not affect the value of a Wilson loop that wraps around the boundary of D_{NS}. The answer one gets this way does depend on the choice of D_{NS} if 't Hooft loops are present also. Upon changing D_{NS} to D'_{NS}, one gets in the gauge theory, in the presence of an 't Hooft loop on C_{RR}, a change in the Wilson loop expectation value given by the same formula as in 5.7. That is because, if the linking number is nonzero, the liftings of the $SU(N)/\mathbb{Z}_N$ bundle to $SU(N)$ on D_{NS} and D'_{NS} disagree on their common boundary.

Of course, different points of view are possible about the expectation value of a product of Wilson and 't Hooft loops on $\mathbf{S}^4$. The traditional point of view is to reject such correlation functions on the grounds that they are not well-defined as complex numbers. If one chooses to consider such correlation functions, we have shown how they must be interpreted, and more specifically we have shown that they have the same meaning whether considered in gauge theory on $\mathbf{S}^4$ or in string theory on $AdS_5 \times \mathbf{S}^5$.

Acknowledgments

This work was supported in part by NSF Grant PHY-9513835. I would like to thank P. Deligne, D. Freed, O. Ganor, M. Hopkins, N. Seiberg, and S. Sethi for discussions.

References

[1] J. Maldacena, *The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, *Gauge theory correlators from non-critical string theory, Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti de Sitter space and holography, Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002].

[4] E. Witten, *Baryons and branes in anti de Sitter space, JHEP* **07** (1998) 006 [hep-th/9805112].

[5] D.J. Gross and H. Ooguri, *Aspects of large N gauge theory dynamics as seen by string theory, Phys. Rev.* **D 58** (1998) 106002 [hep-th/9805129].

JHEP12(1998)012

[6] G. 't Hooft, *On the phase transition towards permanent quark confinement,* *Nucl. Phys.* **B 138** (1978) 1; *A property of electric and magnetic flux in nonabelian gauge theories,* *Nucl. Phys.* **B 153** (1979) 141.

[7] O. Aharony and E. Witten, *Anti de Sitter space and the center of the gauge group,* *JHEP* **11** (1998) 018 [`hep-th/9807205`].

[8] E. Witten, *Fivebrane effective action in M-theory,* *J. Geom. Phys.* **22** (1997) 103 [`hep-th/9610234`].

[9] J. Cheeger and J. Simons, *Differential characters and geometric invariants,* in *Geometry and topology*, J. Alexander and A. Harer eds., Lecture Notes in Mathematics vol. 1167, Springer-Verlag 1985.

[10] A. Schwarz, *The partition function of degenerate quadratic functional and Ray-Singer invariants,* *Lett. Math. Phys.* **2** (1978) 247.

[11] S.W. Hawking and D. Page, *Thermodynamics of black holes in anti de Sitter space,* *Comm. Math. Phys.* **87** (1983) 577.

[12] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,* *Adv. Theor. Math. Phys.* **2** (1998) 505 [`hep-th/9803131`].

[13] C. Böhm, *Noncompact cohomogeneity one Einstein manifolds,* preprint.

[14] M. Rangamani, unpublished.

[15] C. Vafa, E. Witten, *A strong coupling test of S-duality,* *Nucl. Phys.* **B 431** (1994) 3 [`hep-th/9408074`].

[16] D. Friedan and S. Shenker, *The analytic geometry of two-dimensional conformal field theory,* *Nucl. Phys.* **B 281** (1987) 509.

[17] E. Verlinde, *Fusion rules and modular transformations in $2-D$ conformal field theory,* *Nucl. Phys.* **B 300** (1988) 360.

[18] E. Witten, *Quantum field theory and the Jones polynomial,* *Comm. Math. Phys.* **121** (1989) 351.

[19] S. Elitzur, G. Moor, A. Schwimmer, and N. Seiberg, *Remarks on the canonical quantization of Chern-Simons-Witten theory,* *Nucl. Phys.* **B 326** (1989) 108.

[20] S. Axelrod, S. Della Pietra, and E. Witten, *Geometric quantization of Chern-Simons gauge theory, J. Diff. Geom.* **33** (1991) 787.

[21] J. Mickelsson, *Kac-Moody groups, topology of the Dirac determinant bundle, and fermionization,* *Comm. Math. Phys.* **110** (1987) 173.

[22] D. Freed, *Classical Chern-Simons theory: part I, Adv. Math.* **113** (1995) 237 [`hep-th/8206021`].

[23] O. Ganor and S. Sethi, *New perspectives on Yang-Mills theories with sixteen supersymmetries,* *JHEP* **01** (1998) 007 [`hep-th/9712071`].

JHEP12(1998)012

[24] J. Maldacena, *Wilson loops in large N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002].

[25] S.-J. Rey and J. Yee, *Macroscopic strings as heavy quarks in large N gauge theories*, hep-th/9803001.

JHEP12(1998)012